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THREE-DIMENSIONAL CRACK ANALYSIS

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Abstract—A method is presented for the stress analysis of plane cracks of any shape in a stressed three-dimensional linear elastic space. The approach utilizes a system of integral equations which is defined over the crack area only. When these equations are solved for the unknown dislocations, all other quantities related to the crack and the space can then be found. The text contains sections concerning equation system derivation, numerical procedures, stress intensity factors, rectangular cracks, and earthquake control.

INTRODUCTION

Three-dimensional crack analysis has not developed as fully as its two-dimensional counterpart because of its greater complexities. Consequently, very few exact analyses have been done, e.g. the one by Sneddon[1]. There are a number of approximate solutions such as the Thresher and Smith paper[2], but they commonly depend on exact results to some extent. These kinds of studies necessarily lack a general and unified approach.

Recently, however, the so-called boundary integral equation method has achieved some vogue in solving three-dimensional elastostatic problems. It can be applied to the problems considered here, but since a mixed boundary value problem over an infinite domain must be addressed, the solution becomes very awkward. It is found, instead, that the basic equations of that method, when suitably manipulated, provide integral expressions for the crack face stresses. The approach is similar in spirit, if not in detail, to the work of Sanders and his students[3] whereby shell cutout problems are solved by means of integral equations which are defined only over the edge of the cutout or over the crack.

When the stress boundary conditions are applied to the integrals, a system of integral equations is formed in which the unknown functions are the dislocations across the crack. Numerical solutions for the stress intensity factors will be provided for the rectangular crack under various loadings, and these will be applied to an idea which has been proposed to control earthquakes.

THE INTEGRAL EQUATIONS

Consider an infinite elastic space with Cartesian coordinates ξ_i. In the notation of Cruse[4], the j-th component of the traction on a surface with unit normal vector $n_k(\boldsymbol{\xi})$ which is caused by a unit point load in the i-th direction at the point $\mathbf{X}$ is:

$$T_{ij}(\boldsymbol{\xi}, \mathbf{X}) = \frac{-K}{r^2}\left[\frac{\partial r}{\partial n}\left(\delta_{ij} + \frac{3r_{,i}r_{,j}}{1-2\nu}\right) - n_j r_{,i} + n_i r_{,j}\right]$$

where

$$K = (1-2\nu)/8\pi(1-\nu),\ r = |\boldsymbol{\xi} - \mathbf{X}|,\ \delta_{ij}$$

is the Kronecker delta, ν is Poisson's ratio, and differentiations are with respect to $\boldsymbol{\xi}$. The corresponding j-th component of the displacement is:

$$U_{ij}(\boldsymbol{\xi}, \mathbf{X}) = \frac{1}{4\pi G r}\left[\frac{(3-4\nu)\delta_{ij}}{4(1-\nu)} + \frac{r_{,i}r_{,j}}{4(1-\nu)}\right].$$

G is the shear modulus. These comprise the Kelvin solution for the unit point load.

Suppose that a body occupies part of this space and is bounded by the surface A. By using the

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Kelvin solution, it follows from the application of the Betti-Rayleigh theorem that:

$$u_i(\mathbf{X}) = -\int_A u_j(\boldsymbol{\xi}) T_{ij}(\boldsymbol{\xi}, \mathbf{X})\,\mathrm{d}A(\boldsymbol{\xi}) + \int_A t_j(\boldsymbol{\xi}) U_{ij}(\boldsymbol{\xi}, \mathbf{X})\,\mathrm{d}A(\boldsymbol{\xi}) \tag{1}$$

for points $\mathbf{X}$ inside A. The vectors $u_j(\boldsymbol{\xi})$ and $t_j(\boldsymbol{\xi})$ are the displacements and tractions, respectively, on the surface.

Let A consist of an external surface and an internal one (of a cavity), and assume that the surface stresses only are specified. It is convenient to subtract out of this stress field the field which is produced by applied stresses on the outer surface in the absence of the cavity. The so-called reduced problem is, therefore, load-free except internally where the originally loads are now modified. Thus, in the limit as A (outer) grows without bound, eqns (1) remain the same in form, while A denotes the inner surface only. It must be kept in mind later that the complete solution is the sum of the reduced problem and the deleted one.

A plane crack of any shape is a special kind of cavity which is characterized by two plane surfaces $A^{\pm}$, which are located at $\xi_3 = 0^{\pm}$. The applied stresses on them are assumed to be symmetrical across $\xi_3 = 0$, i.e. that $t_j^+ = -t_j^-$. It is also noted that $T_{ij}^+ = -T_{ij}^-$ and $U_{ij}^+ = U_{ij}^-$, so that eqn (1) can be rewritten as:

$$u_i(\mathbf{X}) = -\int_{A^+} \Delta u_j(\boldsymbol{\xi}) T_{ij}^+(\boldsymbol{\xi}, \mathbf{X})\,\mathrm{d}A(\boldsymbol{\xi}) \tag{2}$$

where $\Delta u_j = u_j^+ - u_j^-$. The unit normal vector on A^+ is $n_i^+ = -\delta_{i3}$.

Since we are concerned with a stress boundary value problem, the strains must be obtained from (2) and inserted into Hooke's law,

$$\sigma_{ij} = 2G\nu\delta_{ij}u_{k,k}/(1-2\nu) + G(u_{i,j} + u_{j,i}).$$

That is, we must find:

$$\sigma_{\alpha 3} = G(u_{\alpha,3}^+ + u_{3,\alpha}^+),\ \alpha = 1, 2 \tag{3a}$$

and

$$\sigma_{33} = 2G(\nu u_{\alpha,\alpha}^+ + (1-\nu)u_{3,3}^+)/(1-2\nu). \tag{3b}$$

Equation (2) may be differentiated under the integral as long as $\mathbf{X}$ does not fall upon A^+ However, the limits do not exist as $\mathbf{X} \to A^+$, and measures in the form of integrations by parts must be taken before the limit process can proceed. For example,

$$u_{i,\alpha} = -\int_{A^+} \Delta u_j \frac{\partial T_{ij}^+}{\partial X_\alpha}\,\mathrm{d}A = \int_{A^+} \Delta u_j \frac{\partial T_{ij}^+}{\partial \xi_\alpha} = -\int_{A^+} \Delta u_{j,\alpha} T_{ij}^+\,\mathrm{d}A.$$

Thus,

$$u_{i,\alpha}^+ = -\lim_{\substack{\mathbf{X}\to A^+ \\ (X_3 \to 0^+)}} \int_{A^+} \Delta u_{j,\alpha} T_{i,j}^+\,\mathrm{d}A,$$

where

$$T_{ij}^+ = \frac{-K}{r^2}\left[\frac{X_3}{r}\left(\delta_{ij} + \frac{3r_{,i}r_{,j}}{1-2\nu}\right) + \delta_{j3}r_{,i} - \delta_{i3}r_{,j}\right]$$

and $r_{,i} = (\xi_i - X_i)/r$. The other somewhat complex and lengthy strain derivations, as used in (3a) and (3b), can be found in Weaver[9].

The collective results are the desired integral equations:

$$\sigma_{\alpha 3} = \frac{E(1-2\nu)}{16\pi(1-\nu^2)}\int_{A^+}\left[\delta_{\beta\alpha}r_{,\gamma} - \delta_{\alpha\gamma}r_{,\beta} + \frac{3r_{,\alpha}r_{,\beta}r_{,\gamma}}{1-2\nu}\right]\frac{\Delta u_{\beta,\gamma}\,\mathrm{d}A}{r^2} \tag{4a}$$

$$\sigma_{33} = \frac{E}{8\pi(1-\nu^2)} \int\!\!\!\!\!\!-_{A^+} \frac{r_{,\alpha}\Delta u_{3,\alpha}\,\mathrm{d}A}{r^2} \tag{4b}$$

The slash through the integral sign means that the Cauchy principal value of the integral must be taken, and E is Young's Modulus. Because of the assumed loading, the shear problem uncouples from the normal stress problem. These equations are valid for all $\mathbf{X}$ not on the boundary of the crack or on singularities of the applied loading. The latter condition implies that $\Delta u_{i,\alpha}$ are continuous.

THE NUMERICAL PROCEDURE

It is sufficient to outline the numerical solution to eqn (4b) since the same holds for (4a) as well. Write (4b) as

$$\sigma(\mathbf{X}) = K' \int\!\!\!\!\!\!-_{A^+} K_\alpha(\boldsymbol{\xi} - \mathbf{X}) f_{,\alpha}(\boldsymbol{\xi})\,\mathrm{d}A(\boldsymbol{\xi}) \tag{5}$$

where σ is the applied stress, K' is a constant, K_α are the kernel functions, and f is the unknown. Now let eqn (5) be defined over the rectangle $|\xi_1| < C$, $|\xi_2| < B$. In nondimensional form (with bars removed) it becomes:

$$\sigma = \int\!\!\!\!\!\!-_{A^+} K_\alpha f_{,\alpha}\,\mathrm{d}A$$

with

$$\bar{\sigma} = \sigma/K',\ \bar{K}_\alpha = a^2 K_\alpha,\ \bar{f} = f/a, \text{ and}$$

$$\bar{\xi}_\alpha = \xi_\alpha/a. \text{ The } a \text{ is a reference length.}$$

For an approximate solution, A^+ is divided into a mesh of smaller rectangles so that

$$\sigma = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int\!\!\!\!\!\!-_{A_{i,j}} K_\alpha f_{,\alpha}\,\mathrm{d}A. \tag{6}$$

Appropriate forms of f over each $A_{i,j}$ must include two important features, namely that (1) they be differentiable, (or have the proper form if load discontinuities exist). (2) They vanish in the edge meshes like the square root of the normal distance from the crack edge (Kassir and Sih [5]).

Thus, when f is denoted by $f_{i,j}$ at each mesh centroid (X_{1_i}, X_{2_j}) the simplest approximations over the interior meshes appear to be (for continuous loads)

$$f(\xi_1, \xi_2) \sum_{m=i-1}^{i+1} \sum_{n=j-1}^{j+1} a_{mn}(\xi_1, \xi_2) f_{m,n}$$

for which

$$a_{mn}(\xi_1, \xi_2) = \frac{\prod\limits_{\substack{k=i-1\\k\neq m}}^{i+1} (\xi_1 - X_{1_k}) \prod\limits_{\substack{l=j-1\\l\neq n}}^{j+1} (\xi_2 - X_{2_l})}{\prod\limits_{\substack{k=i-1\\k\neq m}}^{i+1} (X_{1_m} - X_{1_k}) \prod\limits_{\substack{l=j-1\\l\neq n}}^{j+1} (X_{2_n} - X_{2_l})}.$$

In the side meshes, but not including the corners, a sample approximation is:

$$f(\xi_1, \xi_2) = \sqrt{(B - \xi_2)} \sum_{m=i-1}^{i+1} \sum_{n=N_2-1}^{N_2} a_{mn}(\xi_1, \xi_2) f_{m,n} / \sqrt{(B - X_{2_n})}$$

with

$$a_{mn}(\xi_1, \xi) = \frac{\prod\limits_{\substack{k=i-1\\k\neq m}}^{i+1} (\xi_1 - X_{1_k}) \prod\limits_{\substack{l=N_2-1\\l\neq n}}^{N_2} (\xi_2 - X_{2_l})}{\sum\limits_{\substack{k=i-1\\k\neq m}}^{i+1} (X_{1_m} - X_{1_k}) \prod\limits_{\substack{l=N_2-1\\l\neq n}}^{N_2} (X_{2_n} - X_{2_l})}.$$

And finally, on one of the corner meshes the approximation is:

$$f(\xi_1, \xi_2) = \sqrt{[(C+\xi_1)(B+\xi_2)]} \sum_{m=1}^{2} \sum_{n=1}^{2} \frac{a_{mn}(\xi_1, \xi_2) f_{m,n}}{\sqrt{[(C+X_{1_m})(B+X_{2_n})]}},$$

where

$$a_{mn}(\xi_1, \xi_2) = \frac{\prod\limits_{\substack{k=1\\k\neq m}}^{2} (\xi_1 - X_{1_k}) \prod\limits_{\substack{l=1\\l\neq n}}^{2} (\xi_2 - X_{2_l})}{\prod\limits_{\substack{k=1\\k\neq m}}^{2} (X_{1_m} - X_{1_k}) \prod\limits_{\substack{l=1\\l\neq n}}^{2} (X_{2_n} - X_{2_l})}$$

These expressions for f are inserted into eqn (6), and the integrations are performed in any convenient manner. Thus, when (X_1, X_2) is made to coincide with each of the mesh centroids, an algebraic system of equations for $f_{m,n}$ is derived, and the overall approximation to $f(\xi_1, \xi_2)$ is established through the computer-aided inversion.

STRESS INTENSITY FACTORS

Stress intensity factors are related to the displacements near the edge of the crack. Kassir and Sih have shown that the two-dimensional equations for the plane strain crack hold for the three-dimensional smoothed edge crack also. Thus from Rice[6],

$$u_3 = \frac{K_I}{2G}\left(\frac{r}{2\pi}\right)^{1/2} \sin\left(\frac{\phi}{2}\right)\left[4(1-\nu) - 2\cos^2\left(\frac{\phi}{2}\right)\right]$$

for the opening mode, where (r, ϕ) is a local polar coordinate system in a plane perpendicular to the edge. On the crack face, $\phi = \pi$ so that

$$\bar{u}_3 = \frac{K_I(\bar{r})^{1/2} 2(1-\nu)}{G(2a\pi)^{1/2}},$$

in nondimensional notation.

Furthermore, in abbreviation

$$\Delta\bar{u}_3 = 2\bar{u}_3 = (\bar{r})^{1/2} \bar{H}_I(\xi_1, \xi_2; \bar{\sigma}_{33} = -1)$$

where $\bar{H}_I$, which is evaluated on the edge, is part of the numerical solution previously outlined. By equating these two expressions for $\bar{u}_3$,

$$\frac{K_I(\xi_1, \xi_2)}{(a\pi)^{1/2}} = \frac{G(2)^{1/2} \bar{H}_I(\xi_1, \xi_2; \bar{\sigma}_{33} = -1)}{4(1-\nu)}.$$

It follows also, that for $\sigma_{33} = -p$ (constant)

$$\frac{K_I(\xi_1, \xi_2)}{p(a\pi)^{1/2}} = (2)^{1/2} \pi \bar{H}_I(\xi_1, \xi_2; \bar{\sigma}_{33} = -1)$$

Note the absense of Poisson's ratio. For the shear problem in which the applied stresses are $\sigma_{13}=p$, and $\sigma_{23}=0$, the more important stress intensity factors are:

$$\frac{K_{II}(\pm C,\xi_2)}{p(a\pi)^{1/2}}=\frac{2^{3/2}\pi\bar{H}_{II}(\pm C,\xi_2;\bar{\sigma}_{13}=1)}{1-2\nu}$$

$$\frac{K_{III}(\xi_1,\pm B)}{p(a\pi)^{1/2}}=\frac{2^{3/2}\pi(1-\nu)\bar{H}_{III}(\xi_1,\pm B;\bar{\sigma}_{13}=1)}{1-2\nu}.$$

Again, $\bar{H}_{II}$ and $\bar{H}_{III}$ are parts of the numerical solution, and ν does appear because of the different constant in eqns (4a).

RESULTS FOR RECTANGULAR CRACK

The accuracy of the numerical solution could be tested in three different ways. The first was to affirm apparent convergence by using a sequence of finer meshes and to look for obvious irregularities. The second way was to check for the known plane strain results in the center of a long crack. And the third consisted of confirming predictions made by Budiansky[7]. He showed that by applying a three-dimensional form of the J-integral (Budiansky and Rice[8]) to a long crack, $C/B \gg 1$, under uniform tension

$$\sqrt{[\overline{(K_I/p\sqrt{(\pi B)})^2}]}/_{\text{on end}}=1/\sqrt{(2)}$$

where the bar symbolizes the mean square. In the case of uniform shear parallel to the track $(C/B \gg 1)$,

$$\sqrt{\left[\overline{\left(\frac{K_{II}}{p\sqrt{(\pi B)}}\right)^2}+\overline{\left(\frac{K_{III}}{p\sqrt{(\pi B)}}\right)^2}\right]}\Big/_{\text{on end}}=\frac{1}{\sqrt{[2(1-\nu)]}},$$

and for the same shear but $C/B \ll 1$,

$$\sqrt{\left[\overline{\left(\frac{K_{II}}{p\sqrt{(\pi C)}}\right)^2}+\overline{\left(\frac{K_{III}}{p\sqrt{(\pi C)}}\right)^2}\right]}\Big/_{\text{on end}}=\sqrt{\left(\frac{1-\nu}{2}\right)}.$$

In the actual computations, advantage was taken of the double symmetry (or antisymmetry) in each problem, and the shorter crack dimension was uniformly divided. The meshes in the longer

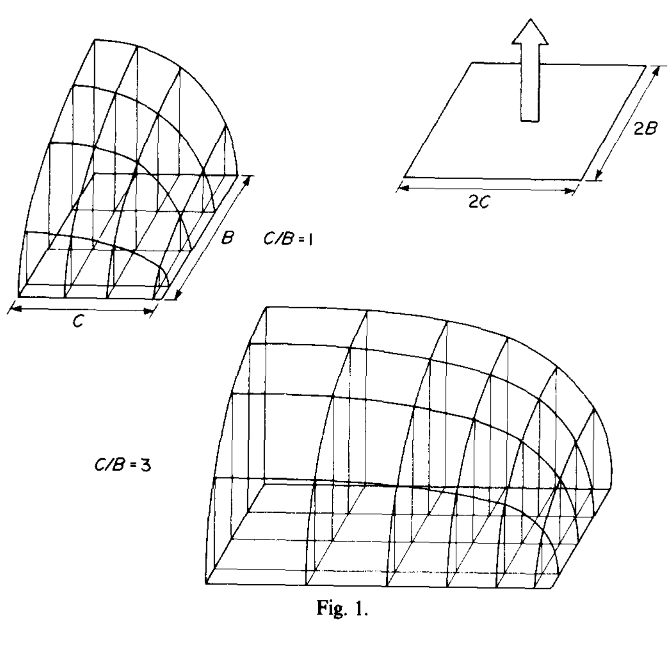

Fig. 1.

direction $\Delta\xi_{1_i}$, say, were made nonuniform with $\Delta\xi_{1_i} = \lambda\,\Delta\xi_{1_{i+1}}$ such that the corner meshes were square and λ did not exceed 1/2.

In the opening mode, 100 unknowns (or less) distributed over the quarter crack were found sufficient to satisfy the above checks with 1% error. Figure 1 shows the displacements for $C/B = 1$ and, typically, $C/B = 3$. Figure 2 is a plot of $K_I/p\sqrt{(\pi B)}$ against position on the shorter side while Fig. 3 is the same but on the other side. It is evident from these that little chance occurs when C/B is larger than 3. Furthermore, end effects extending along the crack on the order of the width can be seen in Fig. 3. The central zone strain field is two-dimensional, and $K_I/p\sqrt{(\pi B)} = 1$ as expected.

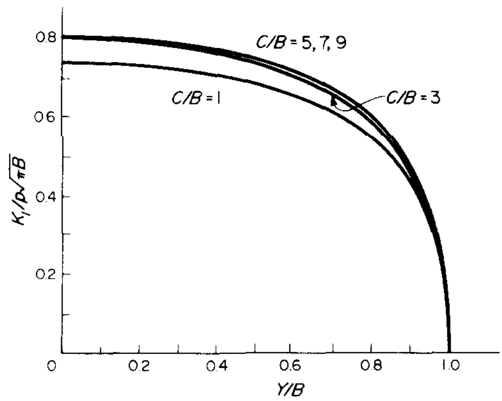

Fig. 2.

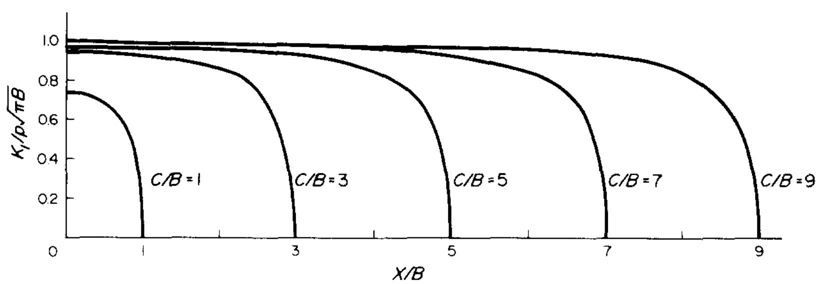

Fig. 3.

When the load was shear, a total of 111 unknowns were needed to bring the end root mean squares to within 1% of their predicted values. Generally the dominant response under shear is the displacement parallel to it. Transverse displacements never exceeded 1/10 of these, and usually they were much less. Qualitatively, the results are similar to the opening mode case. Figures 4–7 show K_{II} and K_{III} where they were larger on any given side.

Plots of u_1 are not given because they are nearly identical to those in Fig. 1.

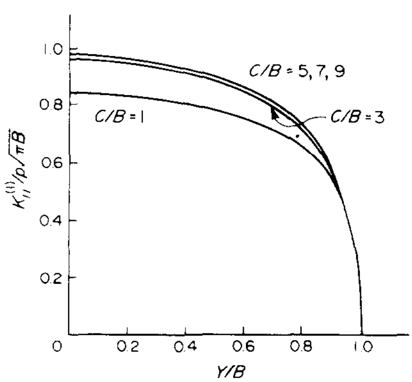

Fig. 4.

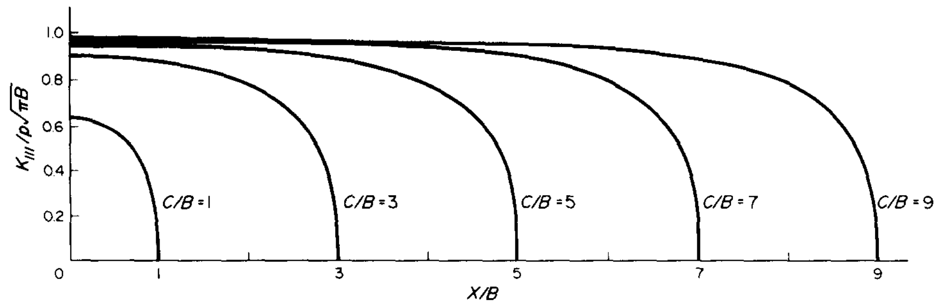

Fig. 5.

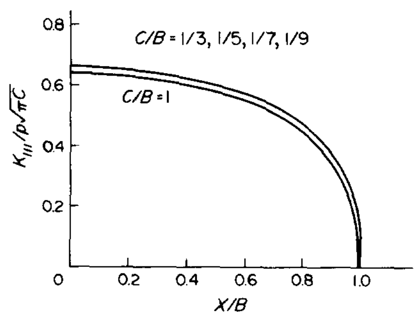

Fig. 6.

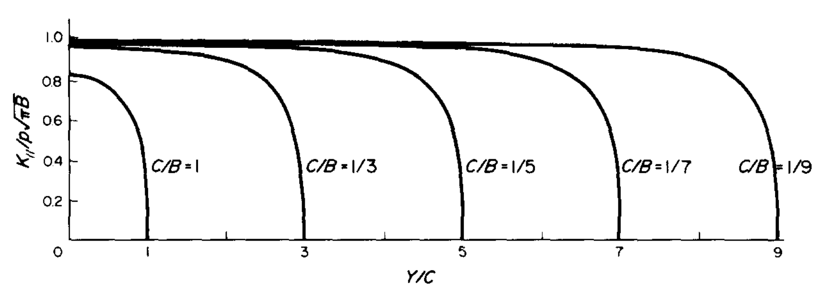

Fig. 7.

EARTHQUAKE CONTROL

The preceeding methods can be applied quantitatively to the study of injecting water into an earthquake fault which would be done in order to induce controlled slip. In this way, accumulated strain energy is carefully released, and the earthquake danger is averted.

The fault zone is modeled as a rectangular crack (as above) under the action of applied shear stress, the tectonic stress. The full space is used for convenience, but it can be seen that there are no shear stresses on the perpendicular plane (earth's surface) which divides the crack and space in two.

Injected water has the effect of reducing the fault's strength so that slippage eventually occurs, and a quasi-static crack (assumed rectangular) begins to grow. The depth, or width, of it is held fixed, and the relationship between the stresses and geometry is sought whereby the stress

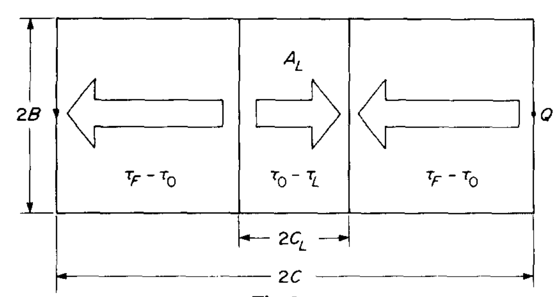

Fig. 8.

singularities at the ends, and of necessity only at the middles of the ends, are suppressed. This is found by determining the stress intensity factors for the different stress systems and summing them to zero.

Figure 8 illustrates the stresses acting upon the crack where τ_0 is the tectonic stress, τ_L is the frictional stress over the water-weakened area A_L, and τ_F is the natural frictional stress. Suppression of the stress singularity is attempted only at the midpoint Q. To ease matters somewhat, this problem is replaced by the superposition of two other problems as indicated in Fig. 9, the first of which has already been solved. Furthermore, the second of these is idealized by using an equivalent concentrated line load, since Q will be reasonably insensitive to the stress distribution over A_L if the crack is not too short. It also eliminates C_L as a parameter. In this last problem, the displacement singularity $u_1 = 2(1-\nu^2)P\ln|\xi_1|/\pi E$ must be appended to the approximate solution given previously, and the integral equations are otherwise homogeneous.

Figures 10–12 show the stress intensity factors of interest. The conditions for crack arrest are:

$$K_{II}^{(1)}(Q) = K_{II}^{(2)}(Q) \quad \text{and} \quad K_{II}^{(1)}(Q) = K_{II}^{(3)}(Q).$$

From these it follows that:

$$C/B = F(\omega) \quad \text{where} \quad \omega = \frac{(\tau_F - \tau_L)}{(\tau_F - \tau_0)}\left(\frac{C_L}{B}\right).$$

Thus, the stress drop by water injection, $(\tau_F - \tau_L)$, and the "margin of safety", $(\tau_F - \tau_0)$, are

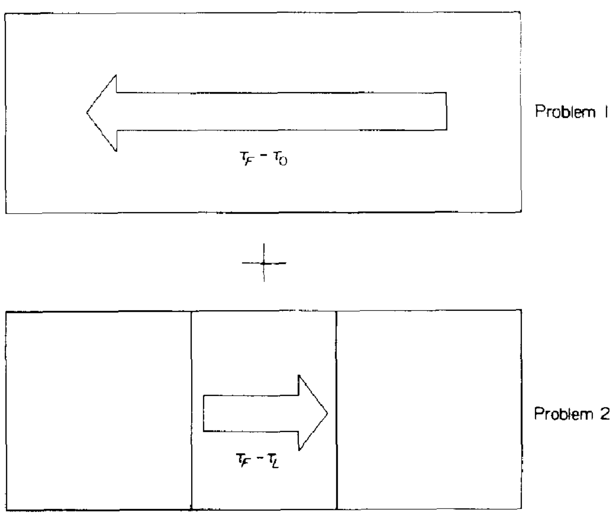

Fig. 9.

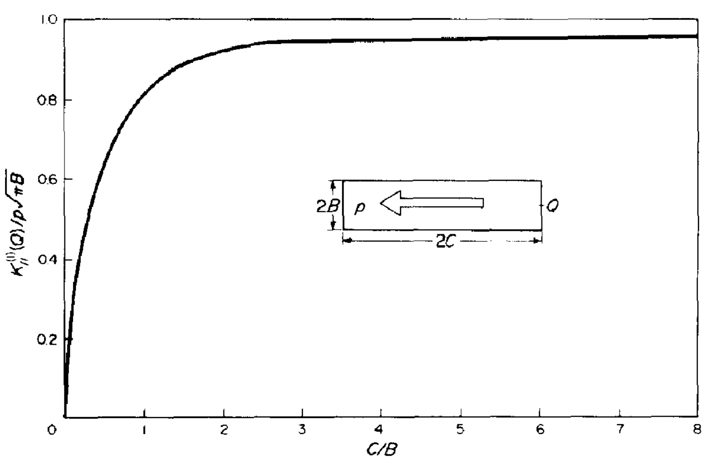

Fig. 10.

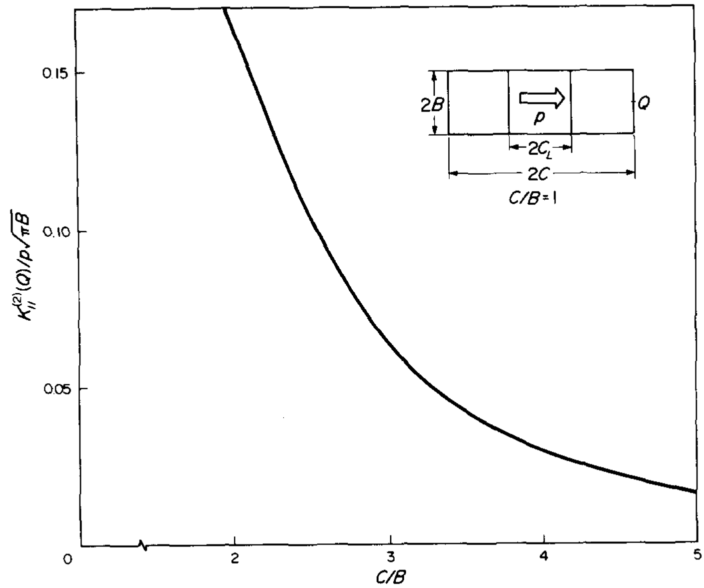

Fig. 11.

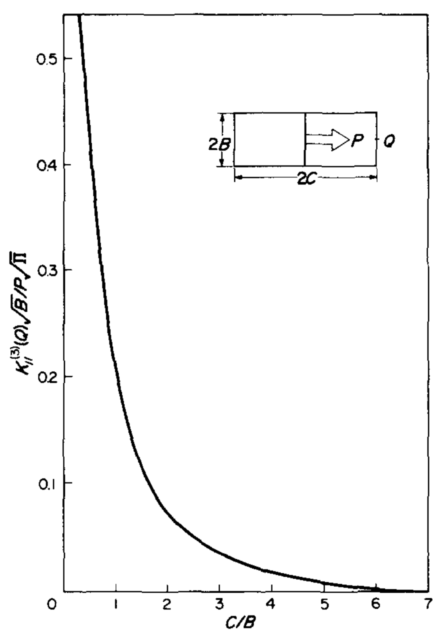

Fig. 12.

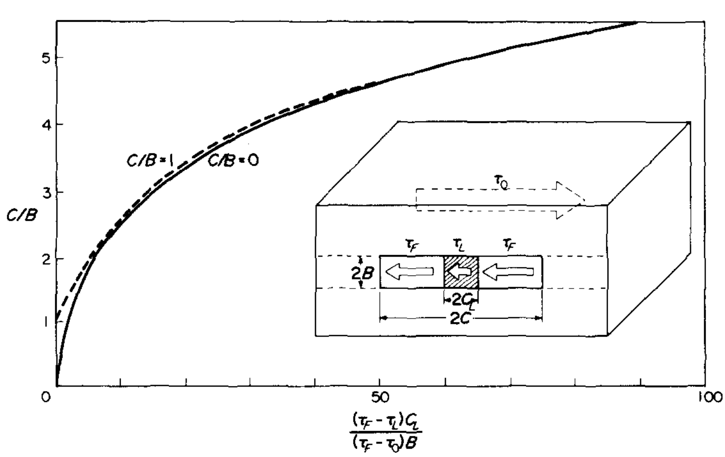

Fig. 13.

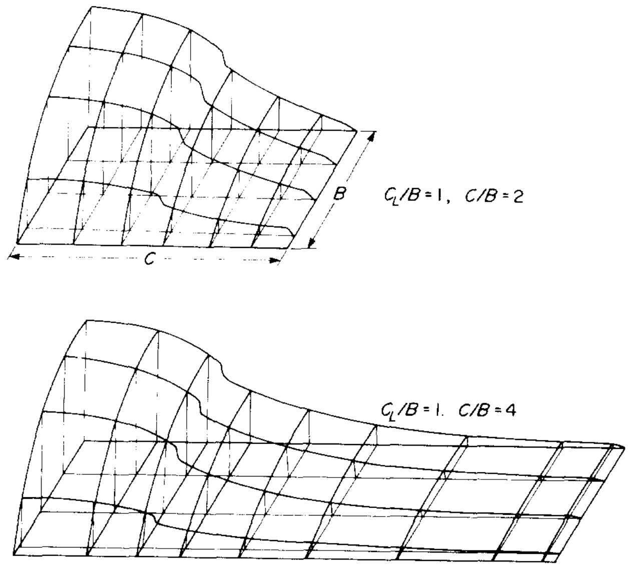

Fig. 14.

explicitly defined. Figure 13 contains the final results for both $C_L/B = 1$ and the line load equivalent, and Fig. 14 shows corresponding dislocations.

As can be seen, C/B is a rather slowly increasing function of ω. This means that the margin of safety must be very small to induce sizeable slippage. In fact, if $\tau_F = 1.01\tau_0$, $\tau_L = 0$ and $C_L/B = 1$, $\omega = 101$ and $C/B \approx 6$, which is not great. Moreover, most dislocation occurs over A_L, so that the phenomenon is really quite local in nature. These conclusions may be of interest to those who would experiment with existing faults.

Further details concerning all aspects of this paper plus the equations for the half space can be found in Weaver[9]. A referee has pointed out the paper by Guidera and Lardner[10] for its complementary contributions.

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